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THE DEVELOPMENT OF FIXED POINT THEORY-Review

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Abstract: *In this paper, we study the observations of progressive results in the fixed point theory. Given a review on some important results from start to at present scenarios.*

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1. INTRODUCTION

The theory of fixed point is one of the most powerful tool of modern mathematical analysis. Theorem concerning the existence and properties of fixed points are known as fixed point theorem. Fixed point theory is a beautiful mixture of analysis, topology & geometry which has many applications in various fields such as mathematics engineering, physics, economics, game theory, biology, chemistry, optimization theory and approximation theory etc. Fixed point theory has its own importance and developed tremendously for the last one and half century. The purpose of the present paper is to study the development of fixed point theory

Definition: Let X be a non-empty set. A function $T : X \longrightarrow X$ is called a self map on X . A point $z \in X$ is called a fixed point of a self map $T: X \longrightarrow X$, if $T(z) = z$

For example the function $T : [0,1] \longrightarrow [0,1]$ defined by $T(x) = x^2$ has exactly two fixed points . This function is uniformly continuous on [0, 1]
The function $S : R \longrightarrow R$ defined by $S(x) = x + 1$ has no fixed point in R.

2. HISTORY OF FIXED POINT THEORY

In the 19th century The study of fixed point theory was initiated by Poincare and in 20th century developed by many mathematicians like Brouwer, Schauder, Kakutani, Banach, Kannan ,Tarski , and others.

Brouwer fixed point theorem

In 1912, Brouwer published his famous fixed point theorem. The theorem states that **Theorem 1.** If B is a closed unit ball in R^n and if $T: B \longrightarrow B$ is continuous then T has a fixed point in B .

Remark: The Brouwer's fixed point theorem guarantees the existence of fixed point. But it does not provide any information about the uniqueness and determination of the fixed point. For example, the function $T : [-1,1] \longrightarrow [-1,1]$ defined by $T(x)= x^3$ is continuous and has three fixed points in $[-1,1]$. Many authors have given different proofs to this theorem. Most of them are topological in nature. This theorem is not true in infinite dimensional spaces.

Schauder's fixed point theorem

in 1930 Schauder was given The first fixed point theorem in an infinite dimensional Banach space . The theorem is stated below:

Theorem 2 If $T: B \longrightarrow B$ is a continuous function on a compact, convex subset B of a Banach space X then f has a fixed point.

Remark: The schauder fixed point theorem is very important and has several applications in economics, game theory, approximation theory etc. In the above theorem Schauder imposed a strong condition of compactness on B. Schauder relaxed this condition and established the following classical result

Theorem 3 If B is a closed bounded convex subset of a Banach space X and T: B → B is continuous map such that T(B) is compact, then T has a fixed point.

Tychonoff fixed point theorem

In 1935 The above Schauder's theorem was generalized to locally convex topological vector spaces by Tychonoff is as follows

Theorem 4 If B is a nonempty compact convex subset of a locally convex topological vector space X and T : B → B is a continuous map, then f has a fixed point

Further extension of Tychonoff's theorem was given by Ky Fan

A very interesting useful result in fixed point theory is due to Banach known as the Banach contraction principle

Banach contraction principal

In 1922 Banach proved a classical fixed point theorem which has many applications in the existence and uniqueness problems of differential equations and integral equations. This theorem is also known as the Banach contraction principle.

Theorem 5 If X is a complete metric space and T : X → X is a contraction map, then f has a unique fixed point or T (x) = x has a unique solution.

While Banach principle came in to existence which was considered as one of the fundamental principle in the field of functional analysis.

In this theorem.Banach proved that a contraction mapping in the field of a complete metric space possesses a unique fixed point. Later on it was developed by Kannan

The fixed point theory (as well as Banach contraction principle) has been studied and generalized in different spaces and various fixed point theorem were developed.

Rothe fixed point theorem

In 1937 Rothe gave a fixed point theorem for non self maps

Theorem 6 If T: B → R^n is a continuous map, such that

$$T(\partial B) \subseteq B, \qquad (1)$$

Then T has a fixed point

The famous fixed point theorem for non expansive maps was given by Browder , Kirk and Gohde independently in 1965

Further extensions of iteration process due to Mann , Ishikawa , and Rhoades are worth mentioning.

The contraction, contractive and nonexpansive maps have been further extended to densifying, and 1- set contraction maps in 1969

In 1966, Hartman and Stampacchia gave the following interesting result in variational inequalities.

Theorem 7 If B is a unit ball in R^n and T : B → R^n a continuous function, then there is a y ∈ B such that

$$< Ty, x - y > \geq 0 \qquad (2) \qquad \text{for all } x \in B .$$

In 1969 the following result was given by KyFan commonly known as the best approximation theorem

Theorem 8 If C is a nonempty compact convex subset of a normed linear space X

And T: C → X is a continuous function, then there is a y ∈ C such that

$$|Ty - y| = \inf|x - Ty| \qquad (3) \qquad \text{for all } x \in C.$$

If P is a metric projection onto C, then Pof has a fixed point if and only if (3) holds.

Recall that d(x, C) = inf||x − y|| for all y ∈ C, x ≠ C.

The Ky Fan's theorem has been widely used in approximation theory, fixed point theory, variational inequalities, and other branches of mathematics.

Theorem 9. If T : B → X is a continuous function and one of the following boundary conditions are satisfied, then f has a fixed point. Here B is a closed ball of radius r and center 0 (∂B stands for the boundary of the ball B).

(i) T(∂B) ⊆ B, (Rothe condition)

(ii) $|Tx - x|^2 \geq |Tx|^2 - |x|^2$, (Altman's condition)

(iii) If Tx = kx for x ∈ ∂B, then k ≤ 1 (Leray Schauder condition)

(iv) If T : B → X and Ty ≠ y, then the line segment [y, Ty] has at least two elements of B. (Fan's condition).

In this survey we have restricted our presentation to single valued maps only. A vast literature is available for the fixed point theorems of multivalued maps. In 1941 Kakutani gave the following generalization of the Brouwer fixed point theorem to multivalued maps.

Theorem 10 If T is a multivalued map on a closed bounded convex C subset of R^n, such that T is upper semicontinuous with nonempty closed convex values, then F has a fixed point. Recall that x is a fixed point of T if x ∈ T x.

The fixed point theory of multivalued maps is useful in economics, game theory and minimax theory. An important application of Kakutani fixed point theorem was made by Nash in the proof of existence of equilibrium for a finite game. Other applications of fixed point theorem of multivalued mapping are in mathematical programming, control theory and theory of differential equations.

Popa introduced implicit functions which are proving fruitful due to their unifying power besides admitting new contraction conditions.

The most recent result for implicit functions is due to Javid Ali and M. Imdad. They introduce an implicit function to prove their results because of their versatility of deducing several contraction conditions in one go. Some new forms of implicit relations are also introduced recently .

APPLICATIONS TO FIXED POINT THEOREM

There are so many applications of fixed point theorems. Some of the applications are as follows:

Integral equations: These equations occur in applied mathematics, engineering and mathematical physics. They also arise as representation formulas in the solution of differential equations.

The Method of Successive Approximations: This method is very useful in determining solutions of integral, differential and algebraic equations.

Conclusion

The fixed point theory from Poincare and Brouwer's theorem to KyFan theorem and fixed point theorems of variational inequalities have been briefly presented. implicit functions are used in engineering, economics, game theory, and the other applied sciences.

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