International Seminar On Non-Conventional Energy Sources for Sustainable Development of Rural Areas,
IJAERD- International Journal of Advance Engineering & Research Development
e-ISSN: 2348-4470, p-ISSN:2348-6406
Impact Factor: 4.72, DOI:10.21090/ijaerd

Common Fixed Point Theorem in Complete Metric Space With Weakly Compatible Maps

Jagrithi Chandra[1] & Sanjay Sharma[2]

[1]*Department of Mathematics,Parthivi College of Engg. & Management(C.G.), India*
[2]*Department of Mathematics, Bhilai Institute of Technology,Durg(C.G.)491001, India*

ABSTRACT: *In this paper the concept of implicit relations & complete metric space which generalizes the result of Brian Fisher by a weaker condition such as weakly compatibility instead of compatibility & contractive modulus instead of continuity of maps.*

KEYWORDS: Common fixed point, Complete metric space, Weakly compatible maps, Contractive modulus MATHS SUBJECT CLASSIFICATION: 47H10, 54H25.

1. INTRODUCTION : the study of common fixed point of mappings satisfying contractive type conditions has been very active field of research activity during last three decades. Brian Fisher [1] proved an important common fixed point theorem In 1922m the Polish mathematician, Banach , proved a theorem which ensures under appropriate conditions, the existence & uniqueness of a fixed point. His result is called Banach fixed point theorem or the Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical science & engineering. Many authors have extended, generalized & improved Banach fixed point theorem in different ways. In[2] Jungck introduced more generalized commuting mappings, called compatible mappings which are more general than commuting & weakly commuting mappings.

The concept of the commutatity has generalized in several ways. For this sessa [6] has introduced the concept of weakly commuting and Gerald Jungck 7 Rhoades [4] introduced notion of weakly compatible & showed that compatible maps are weakly compatible but not conversely.

2. PRELIMINARIES :

Definition 2.1:- A sequence $\{x_n\}$ in a metric space (X, d) is said to be convergent to a point $x \in X$, denoted by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} d(x_n, x_m) = 0$.

Definition 2.2:- A sequence $\{x_n\}$ in a metric space (X, d) is said to be Cauchy sequence if $\lim_{t\to\infty}(x_n, x_m) = 0$ for all n, m > t

Definition 2.3:- A metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.4:- Let f & g be two self maps defined on a set, then f & g are said to be weakly compatible if they commute at coincidence points, i.e., if fu = gu for some $u \in X$ then fgu = gfu.

Definition 2.5:- Let f & g be mapping from a metric space (X,d) into itself. The mapping f & g are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$

Definition 2.6:- A pair (f, g) of self-mappings of a metric space is said to be semi-compatible $\lim_{n\to\infty} fgx_n = gx$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = gx_n = x$

Definition 2.7:- Let f & g be two self-maps on a set X. Maps f & g are said to be commuting if fgx = gfx for all $x \in X$

Definition 2.8:- Let f & g be two self-maps on a set X. If fx = gx , for some x in X then x is called coincidence of f & g.

Definition 2.9:- A function $\emptyset : [0,\infty) \to [0,\infty)$ is said to be contractive modulus if $\emptyset : [0,\infty) \to [0,\infty)$ & $\emptyset(t) < t$ for t > 0.

Definition 2.10:- A real valued function $\emptyset$ defined on X $\subseteq$ R is said to be upper semi continuous if $\lim_{n\to\infty} \emptyset(x_n) \le \emptyset(x)$ for every sequence $\{x_n\} \in X$ with

$x_n \to x$ as $n \to \infty$

3. MAIN RESULT

3.1 Implicit Relations :

Let F* be the set of real functions F (t_1, t_2 , ….t_5): $[0,\infty]^5 \to [0, \infty]$ satisfying

(F_1) F is non increasing in variables t_4 & t_5

(F_2) There is an $h_1 > 0$ & $h_2 > 0$ such that $h = h_1h_2 < 1$ & if $u \geq 0$, $v \geq 0$ satisfy

(F_a) $u \leq F(v, u, v, u+v, 0)$then we have $u \leq h_1v$ & if $u \geq 0$, $v \geq 0$ satisfy

(F_b) $u \leq F(v, u, v, 0, u+v)$ then we have $u \leq h_2v$

(F_3) If $u \geq 0$ is such that $u \leq F(0, 0, u, u, 0)$ then $u = 0$

3.2 Fixed Point Theorem:

Let P, Q, S & T are four self mappings of a complete metric space (X, d) satisfying the following conditions

(a) $P(X) \subseteq T(X)$, $Q(X) \subseteq S(X)$

(b) $d(Px, Qy) \leq F(d(Sx, Ty) ; d(Sx, Px) ; d(Sx, Qy) ; d(Ty, Qy) ; d(Px, ty))$
For all x & y in X where $f \in F^*$

Then P, Q, S & T have unique common fixed point z in X. Further z is the unique common fixed point of P & S & of Q & T

Proof: Suppose $x_0 \in X$

Since $P(X) \subset T(X)$, $Q(X) \subset S(X)$

We can choose x_n , x_{n+1}, x_{n+2} such that

$Px_n = Tx_{n+1}$ & $Qx_{n+1} = Sx_{n+2}$, n = 0, 1, 2, 3……….

Using (b) we have

$$
\begin{aligned}
d(Px_n, Qx_{n+1}) &\leq F(d(Sx_n, Tx_{n+1}); d(Sx_n, Px_{n+1}); d(Tx_{n+1}, Qx_{n+1}); d(Sx_n, Qx_{n+1}); d(Px_n, Tx_{n+1})) \\
&= F(d(Qx_{n-1}, Px_n); d(Qx_{n-1}, Px_n); d(Px_n, Qx_{n+1}); d(Qx_{n-1}, Qx_{n+1}); 0) \\
&\leq F(d(Px_n, Qx_{n-1}); d(Px_n, Qx_{n-1}); d(Px_n, Qx_{n+1}); d(Qx_{n-1}, Qx_{n+1}); 0) \\
&\leq F(d(Px_n, Qx_{n-1}); d(Px_n, Qx_{n-1}); d(Px_n, Qx_{n+1}); d(Px_n, Tx_{n-1}); d(Px_n, Qx_{n+1}))
\end{aligned}
$$

Thus by property (F_a)

$d(Px_n , Qx_{n+1}) \leq h_1 d(Px_n, Qx_{n-1})$

Similarly, $d(Qx_{n+1}, Px_n) \leq h_2 d(Px_{n-2}, Qx_{n-1})$

Therefore, $d(Px_n, Qx_{n+1}) \leq h\, d(Px_{n-2}, Qx_{n-1})$

From this we conclude that $d(Px_n, Qx_{n+1}) \leq h^n d(Px_0, Qx_1)$

$d(Qx_{n+1}, Px_{n+2}) \leq h_2h_1 d(Px_0, Qx_1)$ for n=1,2,………..

Since h < 1 the sequence { Px_0, Qx_1, Px_2, ……,Qx_{n-1}, Px_n , Qx_{n+1}……..)is a Cauchy sequence

Since (X, d) is a complete metric space this sequence has a limit z in X the consequences

$\{Px_n\} = \{Tx_{n+1}\}$ & $\{Qx_{n+1}\} = \{Sx_{n+2}\}$ converge to the point z

We suppose that the mapping S is continuous, so that the sequences $\{S^2x_n\}$ & $\{SPx_n\}$ converge to the point Sz. Since P & S are weakly commute, we have

$d(SPx_n, PSx_n) \leq d(Sx_n, Px_n)$

so that the point $\{PSx_n\}$ converges to the point Sz.

Using (b), we have

$d(PSx_n, Qx_{n+1}) \leq F(d(S^2x_n, Tx_{n+1}) d(S^2x_n, PSx_n) d(Tx_{n+1}, Qx_{n+1}); d(S^2x_n, Qx_{n+1});d(PSx_n, Tx_{n+1}))$

By letting $n \to \infty$, we get

$d(Sz, z) \leq F(d(Sz, z); 0; d(Sz, z); d(Sz, z))$

Therefore by property (F_3) , we get d (Sz, z) = 0 i.e., Sz = z

Again by using (b) , we have

$d(Pz, Qx_{n+1}) \leq F(d(Sz, Tx_{n+1}); d(Sz, Pz); d(Tx_{n+1}, Qx_{n+1}); d(Sz, Qx_{n+1}); d(Pz, Tx_{n+1}))$

By latting $n \to \infty$, we get

$d(Pz, z) \leq F(0; d(z, Pz); 0; 0; d(Pz, z))$

Therefore by property (F_3) , we get (Pz, z) =0 i.e., Pz = z

Since P(X)   T(X) there is a point y in X such that Ty = z
Therefore by (b) , we have

$d(z, Qz) = d(Pz, Qy) \leq F(d(Sz, Ty); d(Sz, Pz); d(Ty, Qy); d(Sz, Qy); d(Pz, Tz))$

so that $d(z, Qy) \leq F(0; 0; d(z, Qy); d(z, Qy); 0)$

Therefore by property (F_3), we get $d(z, qy) = 0$ i.e., Qy = z
Since Q & T are weakly commute , we have
$d(Qz, Tz) = d(QTy, TQy) \leq d(Tz, Qy) = 0$
Thus Qz = Tz & so that by (b), we have

$$d(z, Qz) = d(Pz, Qz) \leq F(d(Sz, Tz); d(Sz, Qz); d(Tz, Qz); d(Sz, Qz); d(Pz, Tz))$$
$$= F (d(z, Qz); d(z, z); d(Qz, Qz); d(z, Qz); d(z, Qz))$$
$$= F (d(z, Qz); 0; 0; d(z, Qz); d(z, Qz))$$

Therefore by property (F_3) , we get $d(z, Qz) = 0$ i.e., Qz = z i.e., z = Qz = Tz
Since Sz = Pz = z, we get z = Qz = Tz = Sz = Pz
Thus z is a common fixed point of P, Q , S & T
On the other way the proof is similar if mapping T is continuous.
Now if we consider that the mapping P or Qis continuous, in the similar way we can prove that z is a common fixed point of P, Q, S & T.

REFRENCES:

[1] B.Fisher, Common Fixed Point of Four Mappings, Bull. Inst. of Math. Academia Sinicia 11(1983), 103-113.
[2] G. Jungck, Compatible mappings & common fixed points, Internet. I. Math & Math. Sci. 9(1986), 771-779.
[3] J. Matkowski, Fixed point theorems for mappings with a contractive iterate at a point, Proc. Amer.Math. Soc., 62(2) (1977), 344-348.
[4] G. Jungck & B.E. Rhoades, Fixed point for set valuesd Functions without Continuity, Indian J. Pure Appl., 29(3) (1988), 227-238.
[5] S. Rezapour & R. Hamlbarani, Some notes on the paper: Cone metric spaces and fixed point theorems of contractive mappings, Journal of Mathematical Analysis & Applications, 345(2)(2008), 719-724.
[6] V. Popa, A general fixed point theorem for four weakly compatible mappings, satisfying an Implicit relation, Filomat 19(2005), 45-51.
[7] V. Sriniwas & R. Uma Maheshwar Rao, A Common Fixed Point theorem for Four Selfmaps, The Mathematics Education Vol. XL, No. 2 June(2006), 99-104.
[8] S. Sessa " On a Weak Commutativity Condition of Mapping in a Fixed Point Considerations, Publ. Inst. Math. Debre., 32(1982), 149-153.
[9] M. Aamri & D. El. Moutawakil, Some new common fixed point theorems under strict Contractive conditions, J. Math. Anal. Appl., 270(2002), 181-188.
[10] M.Imdad, Santosh Kumar & M.S. Khan, Remarks on fixed point theorems satisfying Implicit relations, Radovi Math., 11(2002), 135-143.