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Applications of Laplace Transform and Solution for Various Fractional Differential Equations

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Abstract— *Presently, many researchers have demonstrated the utility of fractional calculus in the derivation of particular solutions of a considerably huge number of linear ordinary and partial differential equations of the second and higher orders. Laplace decomposition technique is applied to achieve series solutions of nonlinear fractional differential equation. The method is based mainly upon some general theorems on (explicit) particular solutions of some families of fractional differential equations with the Laplace transform and the expansion coefficients of binomial series. A major advantage of fractional calculus is that it can be considered as a super set of integer-order calculus. Thus, fractional calculus has the potential to achieve what integer-order calculus cannot. It has been suppose that many of the enormous future developments will come from the applications of fractional calculus to different fields. Laplace transform is a very influential mathematical tool applied in various areas of engineering and science. With the increasing complexity of engineering problems, Laplace transforms help in solving complex problems with a very straightforward approach just like the applications of transfer functions to solve ordinary differential equations. It will allow us to transform fractional differential equations into algebraic equations and then by solving these algebraic equations. The unknown function by using the Inverse Laplace Transform can be obtained.*

Keywords **-** *Laplace transform, fractional derivative, differential equation.*

I. INTRODUCTION

A numerical scheme for solving Linear Non-homogenous Fractional Ordinary Differential Equation is based on Bernstein polynomials approximation. The operational matrices of integration, differentiation and multiplication are introduced and utilized to decrease the Linear Non-homogenous Fractional Ordinary Differential Equation problem in order to solve algebraic equations. The technique is common, easy to implement and yields very precise results Laplace transform is yet another operational tool for solving constant coefficients linear differential equations[1].

The method of solution consists of three major steps:

1) The given difficult problem is transformed into an easy equation.
2) This equation is solved by simply algebraic manipulations.
3) The solution of the easy equation is transformed back to get the solution of the given problem.

In this approach, the Laplace transformation decreases the problem of solving a differential equation to an algebraic problem. The third step is made easier by tables, whose responsibility is analogous to that of integral tables in integration. The Laplace transform can be used to solve differential equations. Besides being a different and efficient alternative to dissimilarity of parameters and undetermined coefficients, the Laplace technique is mainly advantageous for input terms that are piecewise-defined, periodic or impulsive [2].

Fractional differential equation is a simplification of ordinary differential equations and integration to arbitrary non integer orders. The source of fractional calculus goes back to Newton and Leibniz in the 17th century. It is broadly and efficiently used to describe many phenomena arising in engineering, physics, economy and science. Current investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation. Fractional differential equations, consequently find several applications in the field of visco-elasticity, feedback amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles, neuron modeling encompassing different branches of physics, chemistry and biological sciences. There have been many outstanding books and monographs obtainable on this field. Most modern and update developments on fractional differential and fractional integro-differential equations with applications relating many different potentially helpful operators of fractional calculus was given by many. Many physical processes come out to display fractional order behavior that may differ with time or space. The fractional calculus has permitted the operations of integration and differentiation to any fractional order. The order may take on any real or imaginary value. Recently theory of fractional differential equations concerned many scientists and mathematicians to work on. The results have been obtained by using fixed point theorems like Picard's, Schauder fixed-point theorem and Banach contraction mapping principle. About the progress of existence theorems for fractional functional differential equations, much contribution exists. Many applications of fractional calculus amount to replacing the time derivative in a given equation by a derivative of fractional order. The outcome of several studies undoubtedly stated that the fractional derivatives seem to arise universally from significant mathematical reasons. Fractional calculus is a field of mathematics study that grows out of the traditional definitions of calculus integral and derivative operators in much the same way fractional exponents is a result of exponents with integer value. The idea of fractional calculus is not new. In 1695 *L'Hospital* asked the question as to the meaning of $d^n y/dx^n$ if $n = 1/2$; that is "what if n is fractional?". *Leibniz* replied that "$d^{1/2}x$ will be equal to $x \sqrt{dx : x}$". It is normally known that integer-order derivatives and integrals have clear physical and geometric interpretations. However, in case of fractional-order integration and differentiation, which correspond to a rapidly growing field both in theory and in applications to real world problems, it is not so[3].

II. Basic Definition of Laplace Transform

Laplace transform and its inverse is defined by:

$$L : f(t) \to F(s) = L\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt \tag{1}$$

$$L^{-1} : F(s) \to f(t) = L^{-1}\{F(s)\}(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)e^{st}\,dt \tag{2}$$

Note that if f(t) $e^{-\sigma_D t} \to 0$ as $t \to \infty$ then the first integral converges for all complex numbers s with real part greater than s_0, and in the second integral we then require that $a > \sigma_0$ whenever the limit exists (as a finite number). When it does, the integral is said to *converge*. If the limit does not exist, the integral is said to *diverge* and there is no Laplace transform defined for *f*. The factor *s* belongs to some domain in the real or complex plane. The *s* will be properly selected to make sure the convergence of the Laplace integral. Mathematically and technically, the domain of *s* is fairly significant. On the other hand, practically, when differential equations are solved, the domain of *s* is routinely ignored. When *s* is complex, we will forever use the notation $s = x + iy$. The character *L* is the *Laplace transformation*, which acts on functions $f = f(t)$ and generates a new function, $F(s)$ =£f(t) .

Convergence

Although the Laplace operator can be applied to many functions, there are some for which the integral does not converge[4].

Example For the function

$$\lim_{\tau\to\infty}\int_0^\tau e^{-st}\, e^{t^2}\,dt = \lim_{\tau\to\infty}\int_0^\tau e^{t^2}\, e^{-st}\,dt = \infty \tag{3}$$

for any selection of the variable *s*, since the integral grows without bound as $\tau \to \infty$.

In order to go beyond the outward aspects of the Laplace transform, it needs to differentiate two special modes of convergence of the Laplace integral. The integral is said to be *absolutely convergent* if

$$\lim_{\tau\to\infty}\int_0^\tau |\, e^{-st} f(t)\,|\,dt \tag{4}$$

exists. If £f(t) does converge absolutely, then

$$\left|\int_\tau^{\tau'} e^{-st} f(t)\,dt\right| \le \int_\tau^{\tau'} |e^{-st} f(t)|\,dt \to 0 \tag{5}$$

as $\tau \to \infty$, for all $\tau' > \tau$.

There is one more type of convergence which has highest mathematical importance. The integral is called to *converge uniformly* for *s* in domain Ω in the complex plane if for any $\epsilon > 0$, there exists some number τ_0 such that if $\tau \geqslant \tau_0$, then

$$\left|\int_\tau^\infty e^{-st} f(t)\,dt\right| < \epsilon \tag{6}$$

for all *s* in Ω. Here, τ_0 can be selected adequately large in order to create the "tail" of the integral randomly small, *independent of s*.

Continuity Requirements

Since we can calculate the Laplace transform for some functions and not others, such as $e(t2)$, we would like to know that there is a large class of functions that do have a Laplace transform. There is such a class once we make a small number of restrictions on the functions that are considerable.

Definition: A function *f* has a jump discontinuity at a point t_0 if both the limits exist (as finite numbers) and $f(t_0-) = f(t_0+)$. Here, $t \to t-$ and $t \to t+$ that means $t \to t_0$ from the left and right, correspondingly (Figure 1).

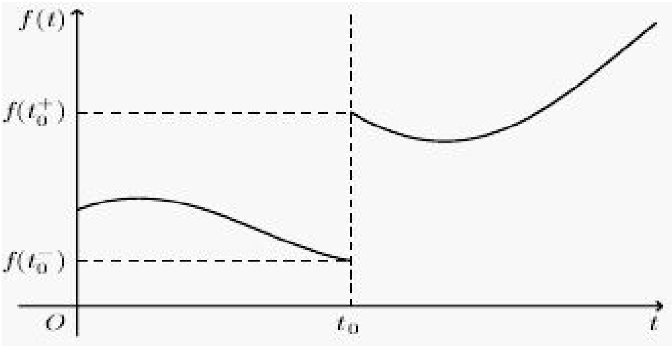

Figure 1

III. Some Important Properties of Laplace Transforms

The Laplace transforms of different functions can be found in the majority of the mathematics and engineering books. Some of the very important properties of Laplace transforms which will be used in its applications are described as follows[5]:

a) Linearity

The Laplace transform of the linear sum of two Laplace transformable functions f(t) + g(t) is specified by

$$L(f(t) + g(t) = F(s) + G(s) \quad (7)$$

b) Differentiation

If the function f(t) is piecewise continuous so that it has a continuous derivative $f^{n-1}(t)$ of order n-1 and a sectional continuous derivative $f^n(t)$ in every finite interval $0 \leqslant t \leqslant T$, then let, f(t) and all its derivatives through $f^{n-1}(t)$ be of exponential order as $t\rightarrow\infty$. Then, the transform of $f^n(t)$ exists when Re(s) > c and has the following form:

$$Lf^n(t) = s^n F(s) - s^{n-1} f(0+) - s^{n-2} f^{(1)}(0+) - \ldots\ldots. - s^{n-1} f^{(n-1)}(0+) \quad (8)$$

c) Time delay

The replacement of t -λ , for the variable t in the transform Lf(t) corresponds to the multiplication of the function F(s) by $e^{-\lambda S}$, that is

$$L\big(f(t-\lambda)\big) = e^{-s\lambda} F(s) \quad (9)$$

d) *t*-Integral Rule: Let *g*(*t*) be of exponential order and continuous for $t \geqslant 0$. Then

$$L(\int_0^t g(x)\,dx = \frac{1}{s} L(g(t)) \quad (10)$$

e) First Shifting Rule: Let *f*(*t*) be of exponential order and $\infty < a < \infty$. Then

$$L\big(e^{at} f(t)\big) = L\big(f(t)\big)|_{s\rightarrow(s-a)}. \quad (11)$$

f) Second Shifting Rule: Let *f*(*t*) and *g*(*t*) be of exponential order and suppose $a \geqslant 0$. Then

$$L\big(f(t-a)H(t-a)\big) = e^{-as} L\big(f(t)\big) \quad (12)$$

$$L\big(g(t)H(t-a)\big) = e^{-as} L\big(g(t+a)\big) \quad (13)$$

g) Periodic Function Rule: Let *f*(*t*) be of exponential order and satisfy *f*(*t* + *P*) = *f*(*t*). Then

$$L\big(f(t)\big) = \frac{\int_0^P f(t)e^{-st}\,dt}{1-e^{-Ps}} \quad (14)$$

h) Convolution Rule: Let *f*(*t*) and *g*(*t*) be of exponential order. Then

$$L\big(f(t)\big)Lg(t)) = L(\int_0^t f(x)g(t-x)dx \quad (15)$$

IV. Application of the Laplace Transform

It is common in engineering education to find the perspective that the Laplace transform is just a theoretical and mathematical concept (outside of the real world) without any application in others areas. The transforms are considered as a tool to make mathematical calculations easier. However, it is important to notice that "frequency domain" is possible appreciate also in the real world and applied in other areas like, for example, economics[6].

"The most popular application of the Laplace transform is in electronic engineering, but it has also been applied to the economic and managerial problems, and most recently, to Materials Requirement Planning (MRP)"

Yu and Grubbstrom (2001)

The article of Grubbstrom (1967) shows the application the Laplace transform to:

• Deterministic Economic Process
• Stochastic Economic Processes

It is pointed out that the technique of the Laplace transform has found a growing number of applications in the fields of physics and technology. For example, there is possibility of solving problems in discounting by this method. Without any loss of common validity, it is made known that the discount factor can always be written in an exponential style which implies that the present value of a "cash-flow" will obtain a very simple form in the Laplace transform terminology. This simplicity holds well for stochastic as well as for deterministic economic processes. It could be also applied to all mathematical simplifications of reality. Grubbstrom (1996) consider a stochastic inventory procedure in which demand is generated by individuals separated by independent stochastic time intervals whereas production takes place in batches of varying sizes at different points in time. The resulting processes are analyzed using the Laplace transform methodology. Then Grubbstrom and Molinder (1994) designed a generalized input-matrix to add in necessities as well as production lead times by means of z-transform in a discrete time representation. The theory is extended to continuous time using the Laplace transform, which enables it to add in the possibility of batch production at finite production rates. Also they developed a fundamental method of how such safety master production plans can be determinate in simple cases using the Laplace transform method[7].

Other application: A telephone or simple intercommunication does not need to be modulated, it is only necessary to have a couple of machines that transform the pressure waves into electric energy, the electric energy is sending by a couple of cables of copper (Cu) and in the receptor side is used a similar machine that convert the variations of electricity in variations of pressure[8]. A microphone and speaker are built in same way. We talk to the microphone by the diaphragm and we make the current in cables variable, by other hand, we put variable current in the cables and the diaphragm produces sound. We can use the space instead of the cable to send signals. There is a lot of difference between send a signal through the cable and to send it by the air. It is like to travel through the flat road or to go through rough road. The air is not as conductive as the copper, otherwise we would be electrocuted. The alternating current (amplitude variable) has the capacity to travel through the space[9]. To transmit a signal from point A to point B trough space we need power and a very high frequency (almost radiating). For lower frequencies we need more power and in the extreme case (continuous current), it does not matter how much power is supplied, it radiates nothing. In certain special frequencies, the higher level of the atmosphere (ionosphere) that act as reflectors and rebound in them is possible to get a signal in large distances. This is "the short wave". If the frequency is too high, it passes across of the large and is lost in the exterior space; if it is too low, does not arrive and it is absorbed by the earth. The problem is that the frequency of the sounds that we are able to hear is between 20Hz and 18 KHz (in people good hearing). If we directly convert the waves of the sound to electricity, they will be of a very low frequency that only can be transmitted by cable. The solution to transmit sounds by the air trough large distances, consist in to send a signal with high enough frequency so that it can radiate, but modifying it in a comparative way to the variations of the sound that we want to send. This frequency is called carrier and the low is modulation[10].

We can modify different things in a carrier and for it different methods of modulation exist like amplitude (AM), frequency (FM), phase (PM), etc. The sounds are variations of pressure in the air and pure sounds exist and composed. The pure sounds consist in only frequency (for example the "beep" of a computer) and are not common in the nature. The majority of the "real" sounds consist in thousand of different frequencies emitted at the same time[11]. This let us to distinguish between a natural sound (rich in resonance) from an artificial (with not many components). The natural sounds like our voice, the music, the noise, etc. they do not consist in a frequency but in a band of frequencies with many fundamentals and other harmonies that produce rebound and resonance in the first.

V. Case Study

Eltayeb. A.M., et.al. "*Laplace Transform Method Solution of Fractional Ordinary Differential Equations*"

The Laplace transform has been applied for solving the fractional ordinary differential equations with constant and variable coefficients. The solutions are expressed in terms of Mittage-Leffller functions, and then written in a compact simplified form. As special case, when the order of the derivative is two the result is simplified to that of second order equation. In this paper they intend to apply Laplace transform technique to solve fractional ordinary differential equations with constant coefficients[12]. To achieve this task, special formulas of Mittage-Leffler function are derived and expressed in terms of elementary functions (power, exponential and error functions), instead of an infinite series. Also special formulations of inverse Laplace transformation are obtained, in terms of Mittage-Leffler functions, which already derived. In obtaining the inverse Laplace transform, some simplified elementary Algebra, relevant to the derived results is used. This work is based on some basic elements of fractional calculus, with special emphasis on the Riemann-Liouville type. For simplicity, they mainly used equations of order (2, 2) with constant coefficients to illustrate this approach. However equation with variable coefficients of order α, where $1<\alpha\leqslant 2$, is considered. The obtained solution agrees with the solution of the classical ordinary differential equation, when $\alpha = 2$. Lastly, it can be concluded that the Laplace transformation method has been effectively applied to find an exact solution of fractional ordinary differential equations, with constant and variable coefficients. Some theorems are introduced; also special formulas of Mittage-Leffler function are derived with their proofs. The method is applied in a direct way without using any assumptions. The results show that the Laplace transformation method needs small size of computations compared to the Adomain decomposition method (ADM), variational iteration method (VIM) and homotopy perturbation method (HPM). The numerical example for equation of variable coefficients shows that the solution in agreement with classical second order equation for It is concluded that the

Laplace transformation method is a powerful, efficient and reliable tool for the solution of fractional linear ordinary differential equations.

VI. Conclusions

To teach the Laplace transform as a separate mathematical topic seems to make it an obstacle for learning. From the survey it has been concluded that it is important to teach simple concepts that ought to have been understood earlier, especially at extreme values, e.g. abstracting an open circuit to an infinite resistance or a short circuit to a zero resistance. Presented survey show the links between theoretical issues and the real circuits have to be made explicitly, something that is also shown in other studies and in the symposium interaction in Lab work - linking the object/event world to the theory/model world. It is common that students learn to make mathematical operations without understanding what they are doing. They just repeat procedures that they have learned to solve the problem.

The Laplace transform is one of the many fields that have teaching contents where it is very easy to disassociate the form and the meaning; the application and understanding of mechanic rules. The idea of some students concerning the Laplace transform is that it is knowledge of strict and unquestionable rules that are applied to problems with just one solution, problems very far of the reality. The disconnection between the application and understanding of procedures in specific situations can be dreadful in engineering education because some students think: mathematics is not necessary understand but it is necessary to know the adequate procedure to solve the problem. For this reason some students use superficial techniques to solve specific circumstances and there is not estrange to notice not motivation and absurd to make just mathematic calculus to pass the subject. It is necessary understand the process: "to go" and "come back" between the formal character, the strict mathematic language and it intuitive and contextual meaning. A person who only can understand a transformation as an action can only make that action, reacting to external indications that give him exact detail about the steps that he has to do. For example, a student that is not able to interpret a situation like a function, with the exception that he has a formula to obtain values, he is restricted to a concept of action of a function. In this case the student cannot make many things with this function, except to evaluate it in specific points and manipulate the formula. It is necessary to mark that mathematics is not disconnected of calculations but it is important do not do the routine calculations without understand the reality. The mathematic is not just a description group of elements.

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